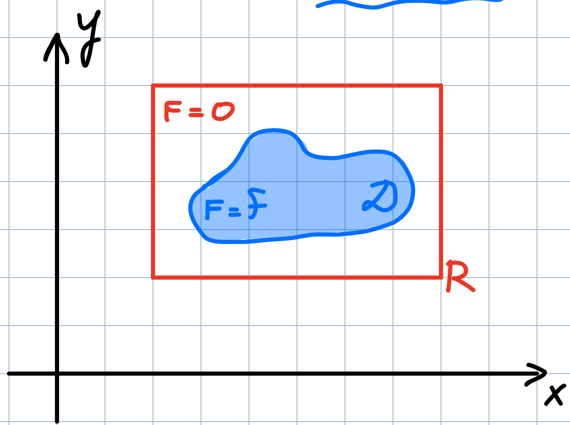
Double integrals over general regions



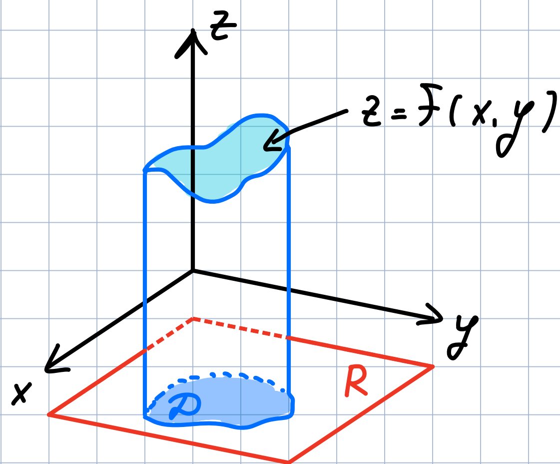
$\mathcal{D}$ - bounded region in $\mathbb{R}^2$,
$f(x,y)$ - function on $\mathcal{D}$

R - rectangle containing $\mathcal{D}$

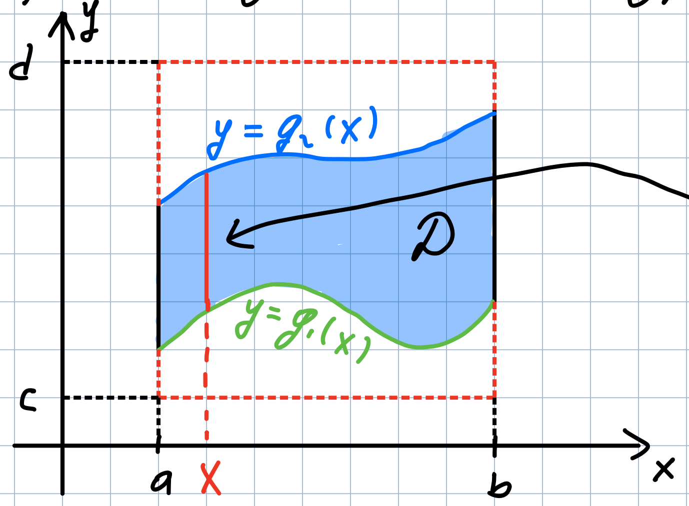
$$\iint_{\mathcal{D}} f(x,y)\,dA \overset{def}{=} \iint_{R} F(x,y)\,dA$$

$$F(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \in \mathcal{D} \\ 0 & \text{if } (x,y) \in R \setminus \mathcal{D} \end{cases}$$

- For $f \geq 0$, $\iint_{\mathcal{D}} f(x,y)\,dA$ = volume of the solid lying above $\mathcal{D}$ and below $z = f(x,y)$



- plane region $\mathcal{D}$ of type I: $\mathcal{D} = \left\{ (x,y) \;\middle|\; \begin{array}{l} a \le x \le b \\ g_1(x) \le y \le g_2(x) \end{array} \right\}$

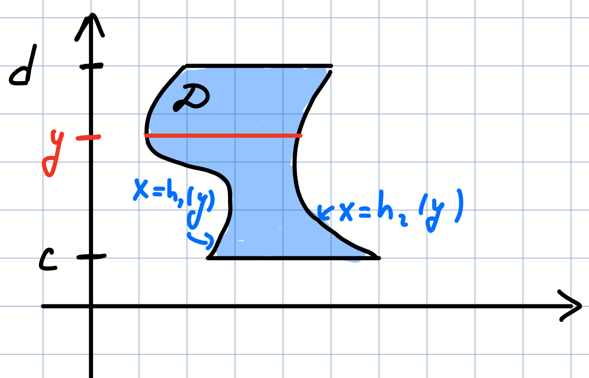


g_1, g_2 - continuous fcts on $[a,b]$

$$\iint_{\mathcal{D}} F(x,y)\,dA = \iint_{R} F(x,y)\,dA = \int_a^b \int_c^d F(x,y)\,dy\,dx$$

$$= \int_a^b \int_{g_1(x)}^{g_2(x)} F(x,y)\,dy\,dx$$ - iterated integral (limits for the inner int. depend on x!)

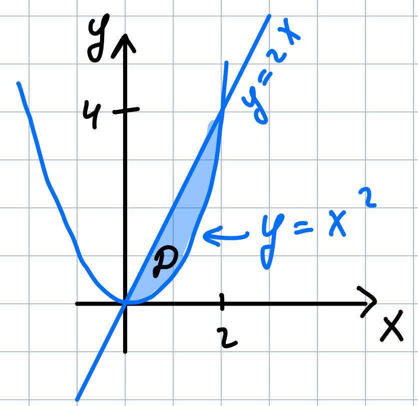
- region of type II: $\mathcal{D} = \left\{ (x,y) \;\middle|\; \begin{array}{l} c \le y \le d \\ h_1(y) \le x \le h_2(y) \end{array} \right\}$



$$\iint_{\mathcal{D}} F(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

Ex: Find the volume of the solid under $z = x^2 + y^2$ and above the region D in xy-plane bounded by $y = 2x$ and $y = x^2$.

Sol:



$$D = \left\{ (x,y) \,\middle|\, \begin{array}{c} 0 \le x \le 2 \\ \underbrace{x^2}_{g_1(x)} \le y \le \underbrace{2x}_{g_2(x)} \end{array} \right\}$$

$$V = \iint_D (x^2+y^2)\, dA = \int_0^2 \underbrace{\int_{x^2}^{2x} (x^2+y^2)\, dy}_{x^2 y + \frac{1}{3}y^3 \Big|_{y=x^2}^{y=2x} = 2x^3 + \frac{8}{3}x^3 - x^4 - \frac{1}{3}x^6} \, dx$$

$$= \int_0^2 \left(-\frac{1}{3}x^6 - x^4 + \frac{14}{3}x^3\right) dx$$

$$= -\frac{1}{21}x^7 - \frac{1}{5}x^5 + \frac{7}{6}x^4 \Big|_{x=0}^{x=2} = -\frac{128}{21} - \frac{32}{5} + \frac{56}{3} = \frac{216}{35}$$

Rmk: Same D can be seen as a type II region:

$$D = \left\{ (x,y) \,\middle|\, \begin{array}{c} 0 \le y \le 4 \\ \frac{y}{2} \le x \le \sqrt{y} \end{array} \right\}$$

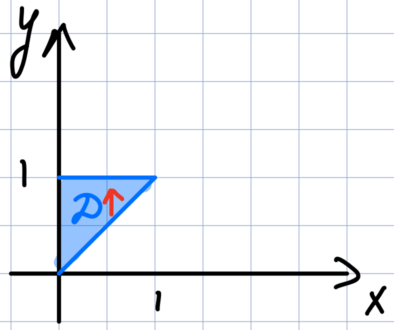
Then

$$V = \int_0^4 \underbrace{\int_{y/2}^{\sqrt{y}} (x^2+y^2)\, dx}_{\frac{x^3}{3} + xy^2 \Big|_{x=y/2}^{x=\sqrt{y}}} \, dy = \int_0^4 \left(\frac{1}{3}y^{3/2} + y^{5/2} - \frac{1}{24}y^3 - \frac{1}{2}y^3\right) dy$$

$$= \frac{2}{15}y^{5/2} + \frac{2}{7}y^{7/2} - \frac{13}{96}y^4 \Big|_{y=0}^{y=4} = \ldots = \frac{216}{35}$$

Ex: Find $\displaystyle\int_0^1\int_x^1 \sin(y^2)\,dy\,dx$

Sol:



$$\int_0^1 \overbrace{\int_x^1 \sin(y^2)\,dy}^{\text{hard!}}\,dx = \iint_D \sin(y^2)\,dA = [\text{view } D \text{ as type II region}]$$

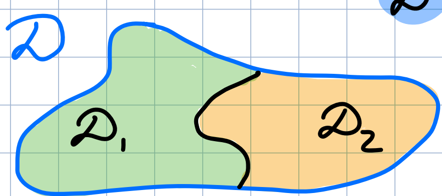
reverse the order of the iterated integral

$$= \int_0^1 \underbrace{\int_0^y \sin(y^2)\,dx}_{x\sin(y^2)\big|_{x=0}^{x=y}}\,dy = \int_0^1 y\sin(y^2)\,dy$$

$$= -\frac{1}{2}\cos(y^2)\Big|_{y=0}^{y=1} = \frac{1}{2}(1-\cos 1)$$

- If $D = D_1 \cup D_2$ with D_1, D_2 not overlaping except perhaps on boundaries, then

$$\iint_D f(x,y)\,dA = \iint_{D_1} f(x,y)\,dA + \iint_{D_2} f(x,y)\,dA$$



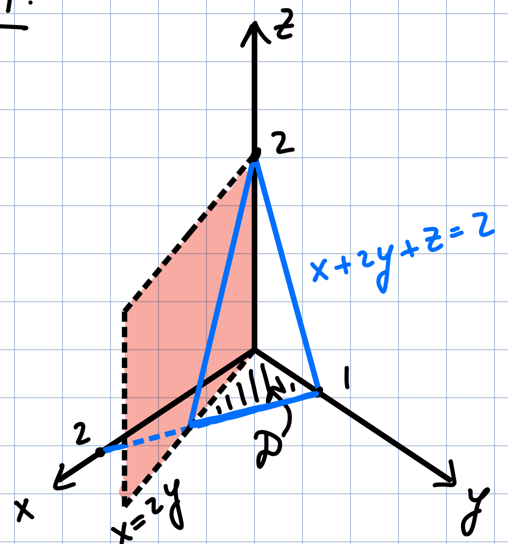
• Average value of f on $\mathcal{D}$:

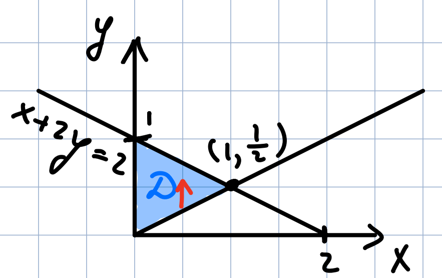
$$f_{avg} = \frac{1}{\text{Area}(\mathcal{D})} \iint_{\mathcal{D}} f(x,y)\, dA$$

Ex: Find the volume of the tetrahidron bounded by planes

$x + 2y + z = 2$, $x = 2y$, $x = 0$ and $z = 0$.

Sol:





$x + 2y + z = 2 \Rightarrow$

$\boxed{z = 2 - x - 2y}$

$$V = \iint_{\mathcal{D}} (2 - x - 2y)\, dA$$

$$= \int_0^1 \underbrace{\int_{\frac{x}{2}}^{1-\frac{x}{2}} (2 - x - 2y)\, dy}_{2y - xy - y^2 \Big|_{y=\frac{x}{2}}^{y=1-\frac{x}{2}} = (2-x)(1-x) - \left(1 - x + \frac{x^2}{4}\right) + \frac{x^2}{4}}\, dx = \int_0^1 (1 - 2x + x^2)\, dx$$

$$= x - x^2 + \frac{1}{3}x^3 \Big|_{x=0}^{x=1} = \frac{1}{3}$$